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DIRECTIONAL DERIVATIVES OF LIPSCHITZ FUNCTIONS

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ABSTRACT

Let f be a Lipschitz mapping of a separable Banach space X to a Banach space Y. We observe that the set of points at which f is differentiable in a spanning set of directions but not Gâteaux differentiable is σ-directionally porous. Since Borel σ-directionally porous sets, in addition to being first category sets, are null in Aronszajn's (or, equivalently, in Gaussian) sense, we obtain an alternative proof of the infinite-dimensional generalisation of Rademacher's Theorem (due to Aronszajn) on Gâteaux differentiability of Lipschitz mappings. Better understanding of σ-directionally porous sets leads us to a new version of Rademacher's theorem in infinite dimensional spaces which we show to be stronger then the one obtained by Aronszajn. A more detailed analysis shows that (a stronger version of) our observation follows from a somewhat technical result showing that the behaviour of the slopes $(f(x+t(u+v))-f(x+tv))/t$ as $t \to 0+$ is in some sense independent of v. In particular, this implies that in the case of Lipschitz real valued functions the upper one-sided derivatives coincide with the derivatives defined by Michel and Penot, except for points of a σ-directionally porous set. This has a number of interesting consequences for upper and lower directional derivatives. For example, for all $x \in X$, except those which belong to a σ-directionally porous set, the function

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$v \to \overline{f}(x, v)$ (the upper right derivative of f at x in the direction v) is convex.

1. Introduction

The simple fact that some exceptional sets which arise naturally in the study of (directional) differentiability of Lipschitz functions on separable Banach spaces are σ-porous had been pointed out by the authors in several lectures, but we felt that the corollaries were not strong enough to warrant a publication. (However, the corresponding main idea was used in [NZ].) Several years ago we noted that these exceptional sets are even σ-directionally porous. This, we now believe, was an important observation, since Borel σ-directionally porous sets are not only first category sets (as all σ-porous sets are) but are also Aronszajn null (or, equivalently, Gaussian null). This was the motivation for the study of the structure of σ-directionally porous sets in [PZ], where we also state the result that the upper derivative of a real-valued Lipschitz function on a separable Banach space is a convex function of direction at all points except possibly those belonging to a σ-directionally porous set. Recently, we noticed that an application of the notion of σ-directional porosity gives, in addition to a 'simple' proof of Aronszajn's version of Rademacher's theorem, also several improvements of this theorem. We believe that these results show that the notion of σ-directional porosity is useful and have therefore decided to publish this account of the above observations.

In the meantime, essentially the same main idea was used in [BC]. For example, the result on convexity of directional derivatives mentioned above is stated in [BC] (Theorem 5.2 and Remark 3.1) for the case of a Hilbert space X and proved in the finite-dimensional case. In [BC] one may also find a finite dimensional analogy of the result that have led us to the improvement of Aronszajn's theorem (Corollary 4.11) and the observation that it gives one of the 'simple' proofs of Aronszajn's theorem. (However, as far as we can see, for the improvement the infinite dimensional version is necessary.)

The paper is organised as follows.

In Section 2 we prove that, if f is a Lipschitz mapping of a separable Banach space to a Banach space Y, then the set of points at which f is differentiable in a spanning set of directions but not Gâteaux differentiable is σ-directionally porous (Theorem 2); since this suffices for applications in Section 4, we give a separate proof. We then give an abstract version of this statement (Lemma 3) which shows that the behaviour of the slopes $(f(x+t(u+v))-f(x+tv))/t$ as $t \to 0+$ is in some sense independent of v. It implies, on one side, that Theorem 2 holds even with one-sided derivatives and, on the other side, a result which may be considered as a vector-valued analogue of the results of Section 3. The following sections are independent of each other. Section 3 deals with real-valued Lipschitz functions only and points out that Lemma 3 immediately implies that ordinary and Michel–Penot upper directional derivatives coincide except for points of a σ-directionally porous set. This and known properties of Michel–Penot derivatives then imply the convexity of directional derivatives mentioned above, a certain symmetry result (Theorem 8(iii)) which generalises a known result for Lipschitz functions of one variable, and an improvement of known results on existence of intermediate derivatives. (The last result is stated in [BC] for a Hilbert space and proved in the finite-dimensional case.)

In Section 4 we first discuss a 'simple' proof of Aronszajn's theorem which, if we wish so, may not use Fubini's theorem and so need not distinguish between finite and infinite dimensional spaces. (However, only the proof using Fubini's theorem may be called simple; otherwise we use a somewhat less trivial result of [PZ].) We then prove an abstract statement (Theorem 10) whose weaker form says that a Gâteaux differentiability result holds with exceptional sets belonging to some σ-ideal if and only if directional differentiability result does. After a brief discussion of the notions of σ-ideals suitable for differentiability results we prove the main result of this section, Theorem 12, which improves Aronszajn's version of Rademacher's theorem. Finally, we give examples showing that this is a genuine improvement.

In the last Section 5 we briefly discuss the problem which sets have to belong to exceptional sets for the Rademacher theorem.

In the rest of Introduction we give the basic notation and definitions. Definitions particular to either Section 2 or 3 are given there.

In the following X will be a real Banach space. We say that a mapping of an open subset $G \subset X$ to a Banach space Y is Lipschitz, if the number

$$\mathrm{Lip}(f) := \sup\left\{\frac{\|f(y)-f(x)\|}{\|x-y\|} : x, y \in G\right\}$$

is finite.

If f is a mapping from X to a Banach space Y and $x, v \in X$, then we consider the directional derivative $f'(x,v)$ and the one-sided (right) directional derivative

$f'_+(x,v)$ defined by

$$f'(x,v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}, \quad f'_+(x,v) = \lim_{t \to 0+} \frac{f(x+tv) - f(x)}{t}.$$

We will often mention the notion of sets null in Aronszajn's sense. By a remarkable result of Csörnyei [C] (see also [BL]) these sets coincide with Mankiewicz's ([Ma]) "cube null sets" as well as with Gaussian null sets (cf. [Ph]), and for all but Section 4 any of these notions is sufficient. The results of Section 4, however, develop Aronszajn's original idea, and we will therefore speak about Aronszajn null sets throughout the paper.

Now we recall the definitions of "porosity notions" that we need and present basic comments on them. Further information can be found in the survey article [Z2]. Of course, the notion of porosity can be (and has been) defined in arbitrary metric spaces, but since we work in Banach spaces, we will not do it. The notion of directional porosity needs additional structure and could be defined on suitable manifolds; again, we do not see any advantage in presenting a general definition without application. We use the notation $B(x,r)$ for the open ball with center x and radius r.

Definition: Let X be a Banach space, $M \subset X$ and $a \in X$. Then we say that:

(i) M is porous at a if there exists $c > 0$ such that for each $\varepsilon > 0$ there exist $b \in X$ and $r > 0$ such that $\rho(a,b) < \varepsilon$, $M \cap B(b,r) = \emptyset$ and $r > c\rho(a,b)$.

(ii) M is porous at a in direction v if the $b \in X$ from (i) verifying the porosity of M at a can be always found in the form $b = a + tv$, where $t \geq 0$. We say that M is directionally porous at a if there exists $v \in X$ such that M is porous at a in direction v.

(iii) M is porous (porous in direction v, directionally porous) if M is porous (porous in direction v, directionally porous) at each of its points.

(iv) M is σ-porous (σ-porous in direction v, σ-directionally porous) if it is a countable union of porous sets (sets porous in direction v, directionally porous sets).

Clearly every directionally porous (σ-directionally porous) set is also porous (σ-porous) and it is an easy well-known fact that these concepts coincide in finite-dimensional spaces.

The notion of σ-porosity was introduced by Dolzhenko [D] and since then it has been used and investigated by many authors; in some applications other variants of porosity notions are natural (cf. [Z2]).

Every σ-porous set is clearly a first category set and the Lebesgue density theorem easily implies that every σ-porous subset of $\mathbb{R}^n$ is of Lebesgue measure zero. Unfortunately, a σ-porous subset of an infinite-dimensional separable Banach space need not be null in any "natural measure sense". In fact, in [PT] an example of an F_σ σ-porous subset S of a separable Hilbert space H is constructed in such a way that the complement $C := H \setminus S$ intersects any line in a set of null one-dimensional Lebesgue measure (on this line) and therefore C is null in Aronszajn's sense. As we already noted, by [C] this is equivalent to C being null in Gaussian or cube sense. In particular, G is also of Haar measure zero in Christensen sense (see [Ch] or [BL, Chapter 6] for the definition).

On the other hand, every Borel σ-directionally porous subset of a separable Banach space X is null in Aronszajn's sense. This fact is an easy consequence of Aronszajn's theorem, which says that every Lipschitz function on X is Gâteaux differentiable at all points except a set which is Aronszajn null, and the easy observation (cf. [Z1], p. 299) that a set $E \subset X$ is directionally porous at a point $a \in E$ if and only if the distance function $d(x) := \operatorname{dist}(x, E)$ is not Gâteaux differentiable at a. An independent proof of the same fact is contained in [PZ], where also some other facts concerning σ-directionally porous sets can be found. Namely, the following result is proved in [PZ] (Theorem 4.10 and Remark 4.11).

THEOREM 1: *Let X be a separable normed linear space and let $(v_n)_1^\infty$ be a complete sequence in X (i.e., $\overline{\operatorname{span}\{v_1, \dots\}} = X$). Let $A \subset X$ be a Borel σ-directionally porous set. Then we can write $A = \bigcup_{n=1}^\infty (A_n^+ \cup A_n^-)$, where A_n^+, A_n^- are Borel sets σ-porous in directions $v_n, -v_n$, respectively.*

For yet another argument showing that Borel σ-directionally porous sets are Aronszajn null see the proof of the inclusion $\mathcal{C}^* \subset \mathcal{A}$ of Proposition 13.

We finish with an important remark concerning a permanence property of σ-directionally porous sets. It will be used to prove Corollary 6.

Remark 1: Let $F\colon X \to Y$ be a bilipschitz bijection between Banach spaces X, Y. It is obvious that then $F(M)$ is σ-porous whenever $M \subset X$ is σ-porous. If, moreover, F has all one-sided directional derivatives at all points (in particular, if F is Gâteaux smooth), then it is easy to observe that $F(M)$ is σ-directionally porous (and therefore also Aronszajn null) whenever M is σ-directionally porous.

2. Lipschitz mappings

The starting point of our investigation is the following statement. It can be easily deduced from Lemma 3, but because of its independent interest (see its application in Section 4), we give a separate proof.

THEOREM 2: *Let f be a Lipschitz mapping of an open subset G of a separable Banach space X to a Banach space Y. Then the following implication holds at each point $x \in X$ except a σ-directionally porous set:*

If the directional derivative $f'(x,u)$ exists in all directions u from a set $U_x \subset X$ whose linear span is dense in X, then f is Gâteaux differentiable at x.

Proof: Since the closed linear span of the range $f(G)$ is clearly separable, we may suppose without any loss of generality that Y is separable. We may also suppose $\mathrm{Lip}(f) \neq 0$. For any $u, v \in X$, $y, z \in Y$, $\varepsilon, \delta > 0$ denote by $A(u,v;y,z;\varepsilon,\delta)$ the set of those $x \in G$ such that

(i) $\|f(x+tu) - f(x) - ty\| \leq \varepsilon|t|$ and $\|f(x+tv) - f(x) - tz\| \leq \varepsilon|t|$ for all $|t| < \delta$ and

(ii) $\|f(x+t(u+v)) - f(x) - t(y+z)\| > 3\varepsilon|t|$ occurs for arbitrarily small $|t|$.

To show that $A(u,v;y,z;\varepsilon,\delta)$ is directionally porous, it suffices to prove that $B(x+tv,r) \cap A(u,v;y,z;\varepsilon,\delta) = \emptyset$ whenever $x \in A(u,v;y,z;\varepsilon,\delta)$, $|t| < \delta$ has the property from (ii) and $r = \varepsilon|t|/(2\,\mathrm{Lip}(f))$. For this note that for every $q \in B(x+tv,r)$,

$$\begin{aligned}\|f(q+tu) - f(q) - ty\| \geq &\|f(x+t(u+v)) - f(x+tv) - ty\| \\ &- 2\,\mathrm{Lip}(f)\|q - (x+tv)\| \\ > &\|f(x+t(u+v)) - f(x) - t(y+z)\| \\ &- \|f(x+tv) - f(x) - tz\| - \varepsilon|t| \\ > &\varepsilon|t|,\end{aligned}$$

hence $q \notin A(u,v;y,z;\varepsilon,\delta)$ by (i), as required.

Define A as the union of all the sets $A(u,v;y,z;\varepsilon,\delta)$ obtained by choosing u,v from a dense countable subset U of X, y,z from a dense countable subset V of Y, and rational numbers $\varepsilon,\delta > 0$. Then A is σ-directionally porous. For any $x \in X \setminus A$, any $u_0, v_0 \in X$ such that $f'(x,u_0)$ and $f'(x,v_0)$ exist, and any rational $\varepsilon > 0$ we find rational $\delta > 0$ such that $\|f(x+tu_0) - f(x) - tf'(x,u_0)\| \leq \varepsilon|t|/2$ and $\|f(x+tv_0) - f(x) - tf'(x,v_0)\| \leq \varepsilon|t|/2$ for all $|t| < \delta$, and we pick $y,z \in V$ and $u,v \in U$ such that $\|y - f'(x,u_0)\| < \varepsilon/4$, $\|z - f'(x,v_0)\| < \varepsilon/4$, $\|u - u_0\| < \varepsilon/4\,\mathrm{Lip}(f)$ and $\|v - v_0\| < \varepsilon/4\,\mathrm{Lip}(f)$. Observe that (i) holds since, for example,

$$\begin{aligned}\|f(x+tu)-f(x)-ty\| \leq &\|f(x+tu_0)-f(x)-tf'(x,u_0)\| \\ &+\|f(x+tu)-f(x+tu_0)\|+\|ty-tf'(x,u_0)\| \\ \leq &\varepsilon\|t|/2+\varepsilon\|t|/2+\varepsilon\|t|/4=\varepsilon|t|\end{aligned}$$

for all $|t| < \delta$. Since $x \notin A(u,v;y,z;\varepsilon,\delta)$ and (i) holds, we obtain that, for all sufficiently small $|t| > 0$, $\|f(x+t(u+v))-f(x)-t(y+z)\| \leq 3\varepsilon|t|$ and consequently also $\|f(x+t(u_0+v_0))-f(x)-t(f'(x,u_0)+f'(x,v_0))\| \leq \|f(x+t(u+v))-f(x)-t(y+z)\|+\varepsilon|t| \leq 4\varepsilon|t|$. Hence $f'(x,u_0+v_0)$ exists and is equal to $f'(x,u_0)+f'(x,v_0)$. Since $f'(x,su_0)=sf'(x,u_0)$ for all $s \in \mathbb{R}$, this shows that the set of directions of differentiability of f at any point $x \in X \setminus A$ is a linear space. Since this set is also closed (see Lemma 11 for a proof of this well-known fact), the statement follows. ∎

We observe that the proof of Theorem 2 gives a stronger statement, namely that the set of directions of differentiability of f at a point x forms a closed linear subspace of X, with the exception of a σ-directionally porous set of points x. We may ask if a similar statement holds for one-sided differentiability and we may also ask if results of this type carry in some natural sense over to the upper and lower derivatives. Instead of refining the proof, we give the following somewhat technical statement which, as we will see, easily implies such results.

LEMMA 3: *Suppose that f is a Lipschitz mapping of an open subset G of a separable Banach space X to a Banach space Y and that $\Phi\colon Y \to \mathbb{R}$ is Lipschitz. Then there is a σ-directionally porous set $A \subset X$ such that for every $x \in X \setminus A$ and $u \in X$ the set of the limit points, as $t \to 0+$, of the function*

$$t \to \Phi\Big(\frac{f(x+t(u+v))-f(x+tv)}{t}\Big)$$

does not depend on $v \in X$.

Proof: We can suppose $\mathrm{Lip}(\Phi) \neq 0$ and $\mathrm{Lip}(f) \neq 0$. Note also that the above function has only finite limit points since it is clearly bounded for each $u,v \in X$. Now fix directions $u,v,w \in X$, positive numbers α,δ and real (open) intervals I,J such that the α-neighbourhood of I is contained in J, and consider the set M of those $x \in G$ such that

(i) $\Phi\big(\frac{f(x+t(u+v))-f(x+tv)}{t}\big) \notin J$ for all $0 < t < \delta$ and

(ii) $\Phi\big(\frac{f(x+t(u+w))-f(x+tw)}{t}\big) \in I$ occurs for arbitrarily small $t > 0$.

To show that M is porous in the direction $w-v$, it clearly suffices to prove that $B(x+t(w-v),r) \cap M = \emptyset$ whenever $\delta > t > 0$ has the property from (ii) and

$r = \alpha t/(2\,\mathrm{Lip}(\Phi)\,\mathrm{Lip}(f))$. For this note that for every $z \in B(x+t(w-v),r)$ the distance of $\Phi\left(\frac{f(z+t(u+v))-f(z+tv)}{t}\right)$ and $\Phi\left(\frac{f(x+t(u+w))-f(x+tw)}{t}\right)$ does not exceed $2\,\mathrm{Lip}(\Phi)\,\mathrm{Lip}(f)\left\|\frac{z-(x+t(w-v))}{t}\right\| < \alpha$, so $\Phi\left(\frac{f(z+t(u+v))-f(z+tv)}{t}\right) \in J$ and hence (i) implies that $z \notin M$ as required.

Define A as the union of all the sets M obtained by choosing u, v, w from a dense countable subset V of X, rational numbers α, δ and intervals I, J with rational end-points. Then A is σ-directionally porous. Suppose that $x \in X$, $u_0, v_0, w_0 \in X$ and that $c \in \mathbb{R}$ is a limit point of the function $t \to \Phi\left(\frac{f(x+t(u_0+w_0))-f(x+tw_0)}{t}\right)$ as $t \to 0+$ but not of the function $t \to \Phi\left(\frac{f(x+t(u_0+v_0))-f(x+tv_0)}{t}\right)$. Then we can clearly find rational numbers a, b, α, δ such that $a < c < b$ and

$$\Phi\Big(\frac{f(x+t(u_0+v_0))-f(x+tv_0)}{t}\Big) \notin (a-3\alpha, a+3\alpha) \quad \text{for all } 0 < t < \delta.$$

If we choose $u, v, w \in V$ so that

$$\max(\|u-u_0\|, \|v-v_0\|, \|w-w_0\|) < \alpha/3\,\mathrm{Lip}(f)\,\mathrm{Lip}(\Phi),$$

and put $I := (a-\alpha, b+\alpha)$, $J := (a-2\alpha, b+2\alpha)$, then (i) and (ii) hold, which shows that $x \in A$ as claimed by the Lemma. ∎

Remark 2: Several slightly more general versions of Lemma 3 may be obtained by obvious modifications of its proof: As the completeness of X and Y has not been used, the Lemma holds for normed linear spaces X, Y. Also, f may be assumed to be locally Lipschitz instead of Lipschitz. The target space of Φ could be any separable metric space instead of $\mathbb{R}$. Finally one can show a version of the Lemma in a non-separable X with directions u, v restricted to a fixed separable subspace.

It is also interesting to note that by using the statement with v replaced by $-(u+v)$ one obtains that the function from the Lemma has the same limit points for $t \to 0+$ and for $t \to 0-$.

If Y is separable and we use Lemma 3 with the functions $\Phi_k(z) = \|z - y_k\|$, where y_k are dense in Y, we immediately get

COROLLARY 4: *Suppose that f is a Lipschitz mapping of an open subset G of a separable Banach space X to a separable Banach space Y. Then there is a σ-directionally porous set $A \subset G$ such that for every $x \in G \setminus A$, $u, v \in X$ and $y \in Y$,*

$$\liminf_{t\to 0+}\left\|\frac{f(x+tu)-f(x)}{t} - y\right\| = \liminf_{t\to 0+}\left\|\frac{f(x+t(u+v))-f(x+tv)}{t} - y\right\|$$

and

$$\limsup_{t\to 0+}\left\|\frac{f(x+t(u+v))-f(x+tv)}{t}-y\right\|=\limsup_{t\to 0+}\left\|\frac{f(x+tu)-f(x)}{t}-y\right\|.$$

We use this statement to deduce the following natural generalisation of Theorem 2.

THEOREM 5: *Let f be a Lipschitz mapping of an open subset G of a separable Banach space X to a Banach space Y. Then there is a σ-directionally porous set $A\subset G$ such that for every $x\in G\setminus A$ the set U_x of those directions $u\in X$ at which the (one-sided) derivative $f'_+(x,u)$ exists is a closed linear subspace of X. Moreover, the mapping $u\to f'_+(x,u)$ is linear on U_x.*

Proof: Since the closed linear span of the range $f(G)$ is clearly separable, we may suppose without any loss of generality that Y is a separable space. We show that the statement holds for any $x\in X\setminus A$, where A is the set from Corollary 4. Since f is Lipschitz, the set U_x is closed and the mapping $u\to f'_+(x,u)$ is continuous (even Lipschitz) on U_x. (These well-known facts follow from the "one-sided" analogue of Lemma 11.)

Hence it suffices to show that $f'_+(x,-u)=-f'_+(x,u)$ and $f'_+(x,u+v)=f'_+(x,u)+f'_+(x,v)$ for every $u\in U_x, v\in U_x$.

If $f'_+(x,u)$ exists, we use the second inequality of Corollary 4 with $v=-u$ and $y=f'_+(x,u)$ to infer that

$$\limsup_{t\to 0+}\left\|\frac{f(x)-f(x-tu)}{t}-f'_+(x,u)\right\|=0.$$

Hence $f'_+(x,-u)$ exists and $f'_+(x,-u)=-f'_+(x,u)$.

If $f'_+(x,u)$ and $f'_+(x,v)$ exist, the second inequality of Corollary 4 with $y=f'_+(x,u)$ gives that

$$\limsup_{t\to 0+}\left\|\frac{f(x+t(u+v))-f(x+tv)}{t}-f'_+(x,u)\right\|=0.$$

Hence

$$\begin{aligned}\lim_{t\to 0+}\frac{f(x+t(u+v))-f(x)}{t}=&\lim_{t\to 0+}\frac{f(x+t(u+v))-f(x+tv)}{t}\\&+\lim_{t\to 0+}\frac{f(x+tv)-f(x)}{t}\\=&f'_+(x,u)+f'_+(x,v),\end{aligned}$$

as required. ∎

As an example of application of Theorem 5, we use it, together with the permanency property of σ-directionally porous sets mentioned in Remark 1, to obtain the following result. (See Section 4 for definition of Aronszajn null sets, and note that simple examples [BL, Example 6.35] show that a bilipschitz image of an Aronszajn null set need not be Aronszajn null.)

COROLLARY 6: *Let X, Y be separable Banach spaces, Z be a Banach space with the Radon–Nikodym property and let $f\colon X \to Z$ be a Lipschitz mapping. Denote by N_f the set of all points at which f is not Gâteaux differentiable. Let further $F\colon X \to Y$ be a bilipschitz mapping having all one-sided directional derivatives at all points. Then $F(N_f)$ is Aronszajn null.*

Proof: Denote by $\tilde{N}_f$ the set of all points $x \in X$ at which f has not the one-sided derivative in a direction v_x. Observe that $N_f \setminus \tilde{N}_f$ is σ-directionally porous by Theorem 5 and therefore (cf. Remark 1) also $F(N_f \setminus \tilde{N}_f)$ is σ-directionally porous (and consequently also Aronszajn null). Let N_g be the set of all Gâteaux non-differentiability points of the Lipschitz function $g := f \circ F^{-1}$. Now observe that $F(N_f) \subset N_g \cup F(N_f \setminus \tilde{N}_f)$. Indeed, otherwise there exists a point $x \in \tilde{N}_f$ such that $F(x) \notin N_g$. But this implies that the one-sided derivative in the direction v_x of $f = g \circ F$ at x exists, which is a contradiction. Thus $F(N_f)$ is Aronszajn null. ∎

Remark 3: The statement holds, with the same proof, also in the case when F has all one-sided directional derivatives at all points except a set S such that $F(S)$ is Aronszajn null. It is not difficult to see that each set S which is a countable union of sets with finite Hausdorff dimension has this property.

3. Ordinary and Michel–Penot upper directional derivatives

In this section we show that our results easily apply to the problem of coincidence of ordinary and Michel–Penot upper directional derivatives of Lipschitz real-valued functions and so also to the problem of existence of intermediate derivatives. These notions are defined for real-valued functions only; in this section we therefore consider only such functions.

If f is a real-valued function on a Banach space X, then we define the upper and lower (one-sided) directional derivatives $\overline{f}(x,v)$ and $\underline{f}(x,v)$ by

$$\overline{f}(x,v) = \limsup_{t\to 0+} \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad \underline{f}(x,v) = \liminf_{t\to 0+} \frac{f(x+tv)-f(x)}{t}.$$

We also consider the (upper) Clarke derivative of f at x in the direction v

$$f^0(x,v) = \sup\Big\{\limsup_{n\to\infty}\frac{f(y_n+t_nv)-f(y_n)}{t_n}: y_n \to x,\ t_n \searrow 0\Big\}$$

and the Michel–Penot (upper) derivative ([MP1] and [MP2], where it is called a radial strict derivative) of f at x in the direction v

$$f^{\diamond}(x,v) := \sup_{u\in X}\Big\{\limsup_{t\to 0+}\frac{1}{t}(f(x+tu+tv)-f(x+tu))\Big\}.$$

Clearly

$$\overline{f}(x,v) \le f^{\diamond}(x,v) \le f^0(x,v) \tag{1}$$

and

$$-\underline{f}(x,v) = \overline{(-f)}(x,v). \tag{2}$$

The Michel–Penot directional derivative has the following remarkable properties (see [MP2], Propositions 1.2 and 1.7).

THEOREM MP: *Let X be a Banach space, $x \in X$ and let f be a Lipschitz function on X. Then*

(a) *The function*

$$v \to f^{\diamond}(x,v)$$

is convex and positively homogeneous on X.

(b) *The equality $f^{\diamond}(x,-v) = (-f)^{\diamond}(x,v)$ holds for each direction $v \in X$.*

(c) *If f is Gâteaux differentiable at x then $f^{\diamond}(x,v) = f'(x,v) = \overline{f}(x,v)$ for each $v \in X$.*

Note that analogues of (a) and (b) for the Clarke directional derivative also hold, but that of (c) does not.

Giles and Sciffer [GS1] proved that, for a Lipschitz function on a separable Banach space, at all points $x \in X$ except those which belong to a first category set (i.e., generically), the equality

$$\overline{f}(x,v) = f^0(x,v)$$

holds for each direction $v \in X$. It is well-known that it is not possible to assert that this exceptional set is null in Aronszajn's sense (even in the case $X = \mathbb{R}$).

The just mentioned result of [GS1] and (1) immediately imply that for a Lipschitz function on a separable Banach space, at all points $x \in X$ except those which belong to a first category set A, the equality

$$\overline{f}(x,v) = f^{\diamond}(x,v) \tag{3}$$

holds for each $v \in X$.

Moreover, Aronszajn's infinite-dimensional Rademacher's theorem ([A]) and Theorem MP, (c) immediately imply that the exceptional set A from the above statement is Aronszajn null. As an immediate consequence of Lemma 3 we point out that the following joint strengthening of both these facts concerning the size of the set where (3) holds.

THEOREM 7: *Let G be an open subset of a separable Banach space X and let f be a real Lipschitz function on G. Then, at all points $x \in G$ except a σ-directionally porous set,*

$$\overline{f}(x,v) = f^{\diamond}(x,v)$$

holds for each direction $v \in X$.

Proof: It is sufficient to apply Lemma 3 with $Y = \mathbb{R}$ and $\Psi(y) = y$. ∎

This result, equation (2) and the properties of Michel-Penot derivatives stated in Theorem MP directly imply several interesting facts concerning upper and lower directional derivatives.

THEOREM 8: *Let X be a separable Banach space and let f be a real Lipschitz function on X. Then there exists a σ-directionally porous set $A \subset X$ such that, for all $x \in X \setminus A$, the following assertions hold.*

(i) *The function $v \to \overline{f}(x,v)$ is convex.*
(ii) *The function $v \to \underline{f}(x,v)$ is concave.*
(iii) *The equality $\underline{f}(x,v) = -\overline{f}(x,-v)$ holds for each $v \in X$.*

Note that the symmetric behaviour of Lipschitz functions given by condition (iii) in the case $X = \mathbb{R}$ is well-known; it is an immediate consequence of a result of [EH] on Dini derivatives of monotone functions, see also [T], Example 66.2 and Theorem 73.1.

Theorem 8 (except (iii)) is stated in [BC] in the case X is Hilbert (and proved in the finite-dimensional case) and in [PZ], p. 297 in the general case. Its Baire category version (in which the exceptional set A is a only first category set) follows, of course, directly from the result of Giles and Sciffer mentioned above.

As an immediate consequence of Theorem 8 we obtain a strengthening of existence results for intermediate derivatives in separable spaces. Following [FP] we say that $x^* \in X^*$ is an intermediate derivative of a function $f\colon X \to \mathbb{R}$ at a point $x \in X$ if

$$\underline{f}(x,v) \leq (x^*,v) \leq \overline{f}(x,v) \quad \text{for every } v \in X.$$

Since f clearly has the intermediate derivative at all points at which it is Gâteaux differentiable, the infinite-dimensional Rademacher's theorem [A] implies that every (locally) Lipschitz function on a separable space X has the intermediate derivative at all points except a set P which is null in Aronszajn's sense. This also implies that P is of the first category, since it is not difficult to prove that in this case the set of points of intermediate differentiability of a Lipschitz function is G_δ. Much stronger information is contained in the following result, which is in the case of a Hilbert space X stated also in [BC].

THEOREM 9: *Let f be a Lipschitz function on a separable Banach space X. Then the intermediate derivative of f exists at all points $x \in X$ except those which belong to a σ-directionally porous set.*

Proof: Use Theorem 8 to choose a σ-directionally porous set $P \subset X$ such that, for each $x \in X \setminus P$, $\underline{f}(x,v) = -\overline{f}(x,-v)$ for each $v \in X$ and the function $v \to \overline{f}(x,v)$ is convex. Let $x \in X \setminus P$ be given. Since the function $v \to \overline{f}(x,v)$ is convex, continuous and positively homogeneous, the Hahn–Banach theorem provides us with $x^* \in X^*$ such that $(x^*,v) \leq \overline{f}(x,v)$ for every $v \in X$. Since

$$(x^*,v) = -(x^*,-v) \geq -\overline{f}(x,-v) = \underline{f}(x,v)$$

we see that x^* is an intermediate derivative of f at x. ∎

We should point out that this result goes in somewhat different direction than those of [FP]. These authors introduced the intermediate derivatives to study (mainly) the non-separable situation; their main result shows that the intermediate derivative of a Lipschitz real-valued function on X exists at all points except a first category set provided that X is a subspace of a space Y that contains a dense continuous image of an Asplund space. This condition holds if X is separable, but, as we have seen before Theorem 9, in this case the result is more straightforward. Another intermediate differentiability result, with a "uniform intermediate derivative" and a "σ-globally porous" exceptional set was recently obtained for (possibly non-separable) superreflexive Banach spaces in [Z3] as an easy consequence of a rather deep result of [BJLPS]. Note, however, that the notions of σ-directional porosity and "σ-global porosity" are incomparable, and so are the results of [Z3] and Theorem 9.

4. Stronger versions of Rademacher's theorem in infinite-dimensional spaces

In this section we apply the notion of σ-directional porosity to improve the known results on the size of sets of points of Gâteaux non-differentiability of Lipschitz mappings. The proofs are self-contained except for the use of Theorem 1 and for the proofs of Borel measurability of this set; the former may be found in [PZ], and the latter in any paper containing a version of Rademacher's theorem in infinite-dimensional spaces, for example [A] or [BL, Proof of Theorem 6.42]. We start by pointing out that the statement

$(*)$ Borel σ-directionally porous sets are Aronszajn null

together with Theorem 2 provides us with a simple proof of Aronszajn's version of Rademacher's theorem for which we do not have to distinguish between finite and infinite dimensional spaces:

A Lipschitz mapping of a separable Banach space X to a Banach space Y with the Radon–Nikodym property is Gâteaux differentiable except at points of an Aronszajn null set.

Indeed, for any spanning set $e_k \in X$ we have (by one of the definitions of the Radon–Nikodym property) that the set $E_k = \{x \in X\colon f'(x, e_k) \text{ doesn't exist}\}$ is null on every line in the direction e_k; it is also easy to see that it is Borel (cf. Lemma 11 below). By Theorem 2, f is Gâteaux differentiable except at points of $A \cup \bigcup_k E_k$, where A is a Borel σ-directionally porous set. Since by $(*)$, A is Aronszajn null, the statement follows.

The needed fact $(*)$ immediately follows from the (non-trivial) Theorem 1; note that this proof does not use Fubini's theorem at all. A simpler proof of $(*)$ (without using, of course, Aronszajn's theorem) may be obtained by the argument used to show that $\mathcal{C}^* \subset \mathcal{A}$ (see the proof of Proposition 13); this is based on a lemma of Aronszajn, which is an easy consequence of Fubini's theorem. Of course, $(*)$ also follows directly from the inclusion $\mathcal{C}^* \subset \mathcal{A}$.

In general terms, our goal may be described as an attempt to define a Borel σ-ideal $\mathcal{N}$ of subsets of X (i.e., a hereditary family of Borel subsets of X closed under countable unions), as small as possible, for which the differentiability statement holds. The above approach naturally leads to the following statement saying roughly that the Gâteaux differentiability result holds for $\mathcal{N}$ provided that the directional differentiability result does.

THEOREM 10: *Suppose that X, Y are Banach spaces, X separable. Then for every Lipschitz mapping f of an open subset G of X to Y and every complete sequence (v_n) in X the set*

$$\{x \in G\colon f \text{ is not Gâteaux diferentiable at } x\}$$

belongs to the Borel σ-ideal generated by the sets of the form

$$\{x \in G\colon f'(x, v_n) \text{ does not exist}\} \quad \text{and} \quad \{x \in X\colon h'(x, v_n) \text{ does not exist}\},$$

where $h\colon X \to \mathbb{R}$ is Lipschitz.

Proof: The statement requires us to show that the set N of those $x \in G$ at which f is not Gâteaux differentiable but $f'(x, v_n)$ exists for all n belongs to the σ-ideal described above. By Theorem 2, N is σ-directionally porous, so by Theorem 1 we can write $N = \bigcup_{n=1}^{\infty}(N_n^+ \cup N_n^-)$, where N_n^+, N_n^- are Borel sets σ-porous in directions $v_n, -v_n$, respectively. Let $N_{n,k}^+, N_{n,k}^-$ be sets porous in directions $v_n, -v_n$, respectively, such that $N_n^+ = \bigcup_{k=1}^{\infty} N_{n,k}^+$ and $N_n^- = \bigcup_{k=1}^{\infty} N_{n,k}^-$. Defining $\varphi_{n,k}(x) = \operatorname{dist}(x, N_{n,k}^+)$ and $\psi_{n,k}(x) = \operatorname{dist}(x, N_{n,k}^-)$, we obtain Lipschitz real-valued functions on X such that $N_{n,k}^+ \subset \{x \in X\colon \varphi'_{n,k}(x, v_n) \text{ does not exist}\}$ and $N_{n,k}^- \subset \{x \in X\colon \psi'_{n,k}(x, v_n) \text{ does not exist}\}$, which gives the statement. ∎

Remark 4: If we are satisfied with a weaker result, namely that the set of non-differentiability points of f belongs to the Borel σ-ideal generated by the sets of the form $\{x \in G\colon f'(x, v) \text{ does not exist}\}$ and $\{x \in X\colon h'(x, v) \text{ does not exist}\}$, where $h\colon X \to \mathbb{R}$ is Lipschitz, then we may restate the above proof without recourse to Theorem 1, since the fact that every Borel directionally porous set belongs to this σ-ideal is straightforward.

We now turn our attention to definitions of various classes suitable for showing Gâteaux differentiability results. The basic idea behind these definitions is as follows: First, a notion of sets "small in a direction v" is defined. Second, one of the two possibilities is chosen: A weaker one defines a set to be small if it can be written as a countable union of sets each of which is small in some direction, or a stronger one which defines a set to be small if for every complete sequence (v_n) in X it can be written as a countable union of sets N_n such that N_n is small in direction v_n. There are two exceptions to this construction of which we are aware. The first are Christensen's Haar null sets [Ch], which, however, form a rather vast class of sets and contain all the classes that we define; they are therefore not a suitable candidate for the smallest σ-ideal for which a differentiability result holds. The second are very recent new null sets of Lindenstrauss and Preiss; they are also not a suitable candidate for the smallest σ-ideal for which a Gâteaux differentiability result holds since they contain some very large sets. In fact, every continuous convex function on any space with separable dual is Fréchet differentiable everywhere except a null set in this sense; by the result of [MM] (see [M] for a generalisation) the set of points of Fréchet differentiability may be Aronszajn null, and so the 'new' null sets may have a complement which is Aronszajn null. It is not clear if these null sets contain any class defined by our approach (in particular, the class $\tilde{\mathcal{A}}$); if not, then more optimal versions of differentiability results have to count with them. We start by describing three notions of smallness in a direction; the first of them is the one used by Aronszajn in [A].

Definition: Let X be a separable Banach space and $0 \neq v \in X$ be given. We define the following classes of sets:

(i) $\mathcal{A}(v)$ is the system of all Borel sets $B \subset X$ such that $B \cap (a + \mathbb{R}v)$ is Lebesgue null on each line $a + \mathbb{R}v$, $a \in X$.

(ii) $\mathcal{A}^*(v, \varepsilon)$ is the system of all Borel sets $B \subset X$ such that $B \in \mathcal{A}(w)$ for each w such that $\|w - v\| < \varepsilon$, and $\mathcal{A}^*(v)$ is the system of all sets B such that $B = \bigcup_{k=1}^{\infty} B_k$, where $B_k \in \mathcal{A}^*(v, \varepsilon_k)$ for some $\varepsilon_k > 0$.

(iii) $\tilde{\mathcal{A}}(v, \varepsilon)$ is the system of all Borel sets $B \subset X$ such that $\{t : \varphi(t) \in B\}$ is Lebesgue null whenever $\varphi: \mathbb{R} \to X$ is such that the function $t \to \varphi(t) - tv$ has Lipschitz constant at most ε, and $\tilde{\mathcal{A}}(v)$ is the system of all sets B such that $B = \bigcup_{k=1}^{\infty} B_k$, where $B_k \in \tilde{\mathcal{A}}(v, \varepsilon_k)$ for some $\varepsilon_k > 0$.

Remark 5: The property of φ from the above definition is clearly equivalent to the condition that

$$\left\| \frac{\varphi(s) - \varphi(t)}{s - t} - v \right\| \leq \varepsilon \quad \text{for } s \neq t.$$

This condition is clearly satisfied if φ is Lipschitz and $\|\varphi'(t) - v\| \leq \varepsilon$ for almost all $t \in \mathbb{R}$; moreover, these two conditions are equivalent provided that X has the Radon–Nikodym property.

Further, a simple extension argument shows that if we consider φ defined on an arbitrary closed interval, or even on an arbitrary set (instead of on the whole $\mathbb{R}$), we obtain the same notion of $\tilde{\mathcal{A}}(v, \varepsilon)$.

Clearly, $\mathcal{A}(v) \supset \mathcal{A}^*(v) \supset \tilde{\mathcal{A}}(v)$. We note that $\mathcal{A}(\lambda v) = \mathcal{A}(v)$, $\mathcal{A}^*(\lambda v) = \mathcal{A}^*(v)$, and $\tilde{\mathcal{A}}(\lambda v) = \tilde{\mathcal{A}}(v)$ for $\lambda \neq 0$; the only statement that may require an argument is the last one, which follows by considering the function $t \to \varphi(\lambda t)$.

The definition of the three classes of negligible sets corresponding to the above notions should now be clear. We use the letter $\mathcal{C}$ for the weaker definition (since the basic observation of Christensen [Ch] implies that $\mathcal{C}$ is non-trivial) and $\mathcal{A}$ for the stronger definition, since its idea is due to Aronszajn and $\mathcal{A}$ is the class of Aronszajn null sets.

Definition: Let X be a separable Banach space.

(i) We define $\mathcal{C}$, C^* and $\tilde{\mathcal{C}}$, respectively, as the system of those subsets B of X that can be written as $B = \bigcup_{n=1}^{\infty} B_n$, where each B_n belongs to $\mathcal{A}(v_n)$, $\mathcal{A}^*(v_n)$ or $\tilde{\mathcal{A}}(v_n)$, respectively, for some v_n.

(ii) We define $\mathcal{A}$, $\mathcal{A}^*$ and $\tilde{\mathcal{A}}$, respectively, as the system of those subsets B of X that can be, for every given complete sequence v_n in X, written as $B = \bigcup_{n=1}^{\infty} B_n$, where each B_n belongs to $\mathcal{A}(v_n)$, $\mathcal{A}^*(v_n)$ or $\tilde{\mathcal{A}}(v_n)$, respectively.

Remark 6: The fact that σ-porous subsets of $\mathbb{R}$ are Lebesgue null immediately implies that each Borel set which is σ-directionally porous in a direction v (or $-v$) belongs to $\mathcal{A}(v)$. Therefore Theorem 1 implies that each Borel σ-directionally porous set belongs to $\mathcal{A}$ (i.e. is Aronszajn null); this is what we have used in the proof of Aronszajn's theorem in the beginning of this section. By a slightly more careful argument one may show directly that each Borel set B which is σ-directionally porous in a direction v (or $-v$) belongs to $\mathcal{A}^*(v)$ and even to $\tilde{\mathcal{A}}(v)$. (Indirectly, this follows from Theorem 12.) For this one would write (cf. [PZ]) $B = \bigcup_{n=1}^{\infty} B_n$ where B_n are Borel sets c_n-porous in the direction v (or $-v$) and, to show that $B_n \in \mathcal{A}^*(v,\varepsilon)$, note that every line in direction close to v meets B_n in a porous set, so in a set of measure zero. To show that $B_n \in \tilde{\mathcal{A}}(v,\varepsilon)$ one would replace this argument by noting that for any φ for which the Lipschitz constant of $t \to \varphi(t) - tv$ is small enough the set $\{t : \varphi(t) \in B_n\}$ is porous in $\mathbb{R}$.

Using these remarks one may prove Theorem 12 or its weaker version for $\mathcal{A}^*$ without recourse to Theorem 10.

The first three simple statements of the following Lemma are often used to deduce Borel measurability of sets defined using derivatives. A simple modification of the proof of Theorem 12 would establish that they are sufficient to show our differentiability results for the class $\mathcal{A}^*$. The (also simple) statement (iv) is used to prove Theorem 12 in its full strength.

LEMMA 11: *Let f be a Lipschitz mapping of an open subset G of a Banach space*

X to a Banach space Y. For $x \in G$ and $v \in X$ denote the expression

$$\lim_{\varepsilon \to 0+} \sup \left\{ \left\| \frac{f(x+tv)-f(x)}{t} - \frac{f(x+sv)-f(x)}{s} \right\| : 0 < |t|, |s| < \varepsilon \right\}$$

by $O(f,x,v)$. Then

(i) *$f'(x,v)$ exists if and only if $O(f,x,v) = 0$.*

(ii) *$g(x) := O(f,x,v)$ is a Borel measurable function on G, for each $v \in X$.*

(iii) *$h(v) := O(f,x,v)$ is Lipschitz on X, for each $x \in G$.*

(iv) *If φ: $\mathbb{R} \to X$, $r \in \mathbb{R}$ and the mapping ψ: $t \to \varphi(t) - tv$ has Lipschitz constant strictly less than $O(f,\varphi(r),v)/4\,\mathrm{Lip}(f)$, then the mapping $f \circ \varphi$ is not differentiable at r.*

Proof: The statement (i) is obvious, (ii) follows easily from the fact that, in the definition of $O(f,x,v)$, it is clearly sufficient to consider ε of the form $\varepsilon = 1/n, n \in \mathbb{N}$ and rational t, s, and for (iii) we simply estimate $|h(u)-h(v)| \le 2\,\mathrm{Lip}(f)\|u-v\|$.

To prove (iv), denote $x = \varphi(r)$, find, for any given $\varepsilon > 0$, $0 < |t|, |s| < \varepsilon$ such that

$$\left\| \frac{f(x+tv)-f(x)}{t} - \frac{f(x+sv)-f(x)}{s} \right\| > 3O(f,x,v)/4$$

and estimate

$$\begin{aligned} D :=& \left\| \frac{f\circ\varphi(r+t) - f\circ\varphi(r)}{t} - \frac{f\circ\varphi(r+s) - f\circ\varphi(r)}{s} \right\| \\ \ge& \left\| \frac{f(x+tv)-f(x)}{t} - \frac{f(x+sv)-f(x)}{s} \right\| - \left\| \frac{f(x+tv) - f(\varphi(r+t))}{t} \right\| \\ &- \left\| \frac{f(x+sv) - f(\varphi(r+s))}{s} \right\|. \end{aligned}$$

Since

$$\begin{aligned} \left\| \frac{f(x+tv) - f(\varphi(r+t))}{t} \right\| \le& \frac{1}{|t|} \mathrm{Lip}(f) \|\varphi(r) + tv - \varphi(r+t)\| \\ =& \frac{1}{|t|} \mathrm{Lip}(f) \|\psi(r) - \psi(r+t)\| \\ \le& \mathrm{Lip}(f)\,\mathrm{Lip}(\psi) \le O(f,\varphi(r),v)/4 \end{aligned}$$

and an analogical estimate holds for $\|\frac{f(x+sv)-f(\varphi(r+s))}{s}\|$, we obtain

$$D > 3O(f,\varphi(r),v)/4 - 2O(f,\varphi(r),v)/4 = O(f,\varphi(r),v)/4;$$

so $O(f\circ\varphi, r, 1) \ge O(f,\varphi(r),v)/4$ is strictly positive as required. ∎

It is immediate to see that $\tilde{A}$ is the smallest from the classes defined in Definition 4. (See Proposition 13 for more information on their relations.) We will therefore prove our improvement of the infinite dimensional Rademacher Theorem for this class only.

THEOREM 12: *Let f be a Lipschitz mapping of an open subset G of a separable Banach space X to a Banach space Y with the Radon–Nikodym property. Then f is Gâteaux differentiable at all points of G except those belonging to a set $A \in \tilde{\mathcal{A}}$.*

Proof: We first show that, for any $v \in X \setminus \{0\}$, the set

$$A = \{x \in X : f'(x,v) \text{ does not exist}\}$$

belongs to $\tilde{\mathcal{A}}(v)$. To this end, we define $A_k = \{x : O(f,x,v) > 1/k\}$ and use Lemma 11(i) to reduce the proof to showing that $A_k \in \tilde{\mathcal{A}}(v, 1/4k\,\mathrm{Lip}(f))$. By Lemma 11(ii), A_k are Borel sets. If $\varphi\colon \mathbb{R} \to X$ is such that the function $t \to \varphi(t) - tv_n$ has Lipschitz constant at most $1/4k\,\mathrm{Lip}(f)$, then by Lemma 11(iv) the function $f \circ \varphi$ is non-differentiable at any t for which $\varphi(t) \in A_k$. Hence $\{t \in \mathbb{R} : \varphi(t) \in A_k\}$ is a subset of the set of points at which $f \circ \varphi$ is not differentiable; since $f \circ \varphi$ is Lipschitz and Y has the Radon–Nikodym property, we infer that this set has measure zero as required for showing that $A \in \tilde{\mathcal{A}}(v)$.

If $h\colon X \to \mathbb{R}$ is Lipschitz, then the above proof (and the fact that $\mathbb{R}$ has the Radon–Nikodym property) gives that $\{x \in X : h'(x,v) \text{ does not exist}\}$ also belongs to $\tilde{\mathcal{A}}(v)$. Consequently, for any complete sequence (v_n) in X, Theorem 10 implies that the set N of points of Gâteaux non-differentiability of f can be written as a union $N = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} N_{n,k}$, where $N_{n,k} \in \tilde{\mathcal{A}}(v_n)$. By definition of $\tilde{\mathcal{A}}(v)$, $N_n = \bigcup_{k=1}^{\infty} N_{n,k}$ belongs to $\tilde{\mathcal{A}}(v_n)$. Since $N = \bigcup_{n=1}^{\infty} N_n$, this proves the statement. ∎

Finally, we observe some relations between the classes from Definition 4. Together with the previous Theorem, these results show that the infinite dimensional Rademacher Theorem holds for each of the classes $\mathcal{C}, \mathcal{C}^*, \tilde{\mathcal{C}}, \mathcal{A}, \mathcal{A}^*$ and $\tilde{\mathcal{A}}$. In addition, they show that these results for the classes $\mathcal{C}^*, \mathcal{A}^*$ and $\tilde{\mathcal{A}}$ present different genuine improvements on Aronszajn's Theorem (which covers the case of $\mathcal{A}$). Some questions concerning $\tilde{\mathcal{C}}$ are left open: we do not know whether $\tilde{\mathcal{C}} \subset \mathcal{A}^*$ or $\tilde{\mathcal{C}} = \tilde{\mathcal{A}}$. Another open question is if all six classes coincide in finite dimensional spaces; this is clearly so if $\dim(X) = 1$ and by a recent result of the first named author they coincide also if $\dim(X) = 2$.

PROPOSITION 13: *Let X be a separable Banach space. Then $\mathcal{C} \supset \mathcal{A} \supset \mathcal{C}^* \supset \mathcal{A}^* \supset \tilde{\mathcal{A}}$ and $\mathcal{C}^* \supset \tilde{\mathcal{C}} \supset \tilde{\mathcal{A}}$, and, if $\dim(X) = \infty$, these inclusions, with the possible exception of the last one, are proper. Moreover, $\mathcal{A}^* \setminus \tilde{\mathcal{C}} \neq \emptyset$.*

Proof: With the exception of $\mathcal{A} \supset \mathcal{C}^*$, all the above inclusions follow directly from the definition. To prove that $\mathcal{C}^* \subset \mathcal{A}$, it is clearly sufficient to show that $\mathcal{A}^*(v,\varepsilon) \subset \mathcal{A}$ for each $0 \neq v \in X$ and $\varepsilon > 0$. Let $B \in \mathcal{A}^*(v,\varepsilon)$ and a complete sequence (v_n) be given. Let $u = \sum_{n=1}^m a_n v_n$ be such that $\|u - v\| < \varepsilon$. So $B \in \mathcal{A}(u)$, which by a result of [A] (see also [BL, Proposition 6.29]) implies that it can be written as a countable union of sets from $\mathcal{A}(v_n)$.

To construct the examples, let $I = \{x \in \ell_\infty : 1 \leq x_k \leq 2\}$. We equip I with the topology of pointwise convergence (so it is a compact metrizable space; in fact it is the Hilbert cube) and with the measure μ defined as the product of countably many copies of the Lebesgue measure on $[1,2]$.

Suppose that $\dim(X) = \infty$, $u_0, u_1, \ldots \in X$ and $u_k^* \in X^*$ are such that $u_k^*(u_k) = 1$ and $u_j^*(u_k) = 0$ if $k < j$. We will also require that u_k or u_{2k} are dense in X (in the last example we will require that they are dense in a certain open subset of X). Such u_k, u_k^* may be easily constructed by induction. We also define $c_0 = 1$ and choose recursively $c_k > 0$ so that

$$c_k(|u_j^*(u_k)| + \|u_k\| + 1) \leq 2^{j-k-1} c_j \quad \text{for } 0 \leq j < k. \tag{4}$$

Using (4) for $j = 0$, we see that $c_k \|u_k\| \leq 2^{-k}$ and we infer that the formula $F(t) = \sum_{k=0}^\infty c_k t_{k+1} u_k$ defines a linear mapping of ℓ_∞ to X. We show that it is injective. Indeed, if $t \in \ell_\infty$, $t \neq 0$ and $F(t) = 0$, we choose j so that $|t_{j+1}| > \|t\|_\infty/2$ and infer from $u_j^*(F(t)) = 0$ that $c_j t_{j+1} + \sum_{k=j+1}^\infty c_k t_{k+1} u_j^*(u_k) = 0$. But this and (4) give a contradiction by $c_j \|t\|_\infty/2 < c_j |t_{j+1}| \leq \|t\|_\infty \sum_{k=j+1}^\infty c_k |u_j^*(u_k)| \leq c_j \|t\|_\infty/2$.

Since $\sum_k c_k \|u_k\| < \infty$, the series defining F converges uniformly on I, and so the restriction of F to I is continuous. Hence $F(I)$ is a compact convex subset of X which belongs to $\mathcal{C}$ (since for every compact subset C of X there is $0 \neq v \in X$ such that C meets every line in direction v in at most one point) but not to $\mathcal{A}$ provided that u_k are complete (since $\mu\{t \in I : F(t) \in B\} = 0$ whenever $B \in \mathcal{A}(u_k)$ for some k). Our examples will be obtained by using some of the directions u_k to make this well known example slightly non-linear.

For the example of a set from $\mathcal{A}^* \setminus \tilde{\mathcal{C}}$ we just require that u_k be dense in X. Let

$$F_1(x) = \left(\sum_{k=1}^\infty c_k^2 x_k^2\right) u_0 + \sum_{k=1}^\infty c_k x_k u_k = F\left(\sum_{k=1}^\infty c_k^2 x_k^2, x_1, x_2, \ldots\right).$$

Using (4) for $j = 0$, we obtain that F_1 is well-defined for $x \in l_\infty$. It is easy to see that the restriction of F_1 to I is continuous and thus $F_1(I)$ is a compact set.

Note that, if $x, y \in I$ and $0 < t < 1$ are such that $x \neq y$ and

$$tF_1(x) + (1-t)F_1(y) \in F_1(I),$$

then, since F is a linear injection of ℓ_∞ to X,

$$tF_1(x) + (1-t)F_1(y) = F_1(tx + (1-t)y)$$

and consequently

$$\sum_{k=1}^{\infty} c_k^2(tx_k + (1-t)y_k)^2 = t\sum_{k=1}^{\infty} c_k^2 x_k^2 + (1-t)\sum_{k=1}^{\infty} c_k^2 y_k^2.$$

But this is impossible since the function z^2 is strictly convex on $\mathbb{R}$, so

$$c_k^2(tx_k + (1-t)y_k)^2 \leq tc_k^2 x_k^2 + (1-t)c_k^2 y_k^2 \quad \text{for each } k$$

and, since $x \neq y$, there exists k, for which the inequality is strict. We conclude that each line meets $F_1(I)$ in no more than two points, hence $F_1(I) \in \mathcal{A}^*$.

Suppose now that $F_1(I) \in \tilde{\mathcal{C}}$, hence $F_1(I) = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} B_{n,m}$, where $B_{n,m} \in \tilde{\mathcal{A}}(v_n, \varepsilon_{n,m})$. We show that $\mu(F_1^{-1}(B_{n,m})) = 0$ for each n, m. For any given m, n we find j such that $c_j\|u_0\| < \varepsilon_{n,m}/8$ and $\|u_j - v_n\| < \varepsilon_{n,m}/2$. Let e_j be the j-th member of the canonical basis of l_∞. For any $x \in I$ the mapping $\varphi_x\colon t \to F_1(x + te_j)/c_j$ is Lipschitz on $[1-x_j, 2-x_j]$ and satisfies $\|\varphi_x'(t) - v_n\| = \|2c_j(x_j + t)u_0 + u_j - v_n\| < \varepsilon_{n,m}$, hence the set $\{t : x + te_j \in F_1^{-1}(B_{n,m})\} = \{t : \varphi_x(t) \in B_{n,m}\}$ is Lebesgue null, and Fubini theorem gives $\mu(F_1^{-1}(B_{n,m})) = 0$ as claimed. But this contradicts $I = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} F_1^{-1}(B_{n,m})$, and we infer that $F_1(I) \notin \tilde{\mathcal{C}}$.

For the example of a set from $\mathcal{A} \setminus \mathcal{C}^*$ we require that u_{2k} be dense in X. Define

$$\begin{aligned} F_2(x) =& \sum_{k=1}^{\infty} c_{2k-1}\xi_k x_1 x_2 \cdots x_k u_{2k-1} + \sum_{k=1}^{\infty} c_{2k} x_k u_{2k} \\ =& F(0, \xi_1 x_1, x_1, \xi_2 x_1 x_2, x_2, \xi_3 x_1 x_2 x_3, x_3, \ldots), \end{aligned}$$

where $0 < \xi_k \leq 1/k!$ are such that $\lim_{j\to\infty}\sum_{k=j}^{\infty} 2^k \xi_k c_{2k-1}\|u_{2k-1}\|/c_{2j} = 0$. Note that, if $x, y \in I$ are such that $x \neq y$ and $tF_2(x) + (1-t)F_2(y) \in F_2(I)$ for infinitely many real t, then the fact that F is a linear injection of ℓ_∞ to X implies that, for each k,

$$(tx_1 + (1-t)y_1)\cdots(tx_k + (1-t)y_k) = tx_1\cdots x_k + (1-t)y_1\cdots y_k$$

for infinitely many t, and so for all t, since both sides are polynomials in t. For y differing from x in more than one coordinate the polynomial on the left has, for sufficiently large k, degree greater that one, so the two sides are not equal. (This is why we defined I using the intervals $[1,2]$.) Hence our x and y may differ in one coordinate only. Conversely, for any $x \in I$, the points $F_2(y)$, where $y \in I$ differs from x in j-th coordinate only form a segment on the line $F_2(x)+\mathbb{R}w_j(x)$, where $w_j(x) = \sum_{k=j}^{\infty} \xi_k x_1 \cdots x_{j-1}x_{j+1} \cdots x_k c_{2k-1}u_{2k-1} + c_{2j}u_{2j}$.

The above analysis shows that the set of lines which contain any fixed point $F_2(x)$, $x \in I$, and meet the set $F_2(I)$ in an infinite set is countable. (In fact, the union of these lines is covered by the lines $F_2(x)+\mathbb{R}w_j(x), j = 1,2,\ldots$.) Hence $F_2(I)$ meets every two dimensional subspace of X in a set of two dimensional Lebesgue measure zero which by [A] (see also [BL, Proposition 6.29]) implies that $F_2(I) \in \mathcal{A}$.

Suppose now that $F_2(I) \in \mathcal{C}^*$, hence $F_2(I) = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} B_{n,m}$, where $B_{n,m} \in \mathcal{A}^*(v_n, \varepsilon_{n,m})$. We show that $\mu(F_2^{-1}(B_{n,m})) = 0$ for each n, m. For any given m, n we find j such that

$$\sum_{k=j}^{\infty} 2^k \xi_k c_{2k-1}\|u_{2k-1}\| < c_{2j}\varepsilon_{n,m}/2 \quad \text{and} \quad \|u_{2j} - v_n\| < \varepsilon_{n,m}/2.$$

For any $x \in I$ we have

$$\|w_j(x) - c_{2j}v_n\| \le \sum_{k=j}^{\infty} 2^k \xi_k c_{2k-1}\|u_{2k-1}\| + c_{2j}\|u_{2j} - v_n\| < c_{2j}\varepsilon_{n,m}.$$

So $\|w_j(x)/c_{2j} - v_n\| < \varepsilon_{n,m}$, which shows that the line $F_2(x) + \mathbb{R}w_j(x)$ meets $B_{n,m}$ in a Lebesgue null set. Hence the set

$$\{t : x + te_j \in F_2^{-1}(B_{n,m})\} = \{t : x + tw_j(x) \in B_{n,m}\}$$

is Lebesgue null, and Fubini theorem gives $\mu(F_2^{-1}(B_{n,m})) = 0$. But this contradicts $I = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} F_2^{-1}(B_{n,m})$, and we infer that $F_2(I) \notin \mathcal{C}^*$.

For our last example, of a set from $\mathcal{C}^* \setminus \mathcal{A}^*$, we choose a closed convex cone $K \subset X$ with nonempty interior for which $X \setminus (K \cup (-K)) \neq \emptyset$. We require that $u_k \in K$ and that u_{2k} be dense in K and define F_3: $I \to X$ by $F_3(x) = \sum_{k=1}^{\infty} c_{2k-1}\xi_k x_1 x_2 \cdots x_k u_{2k-1} + \sum_{k=1}^{\infty} c_{2k}x_k u_{2k}$, where not only the formula but also the ξ_k are as in the previous example. The arguments of the previous example together with the observation that $w_j(x) \in K$ for each j and $x \in I$ then show that $F_3(I)$ is null on every line in direction of any vector v not belonging to $K \cup (-K)$. So $F_3(I) \in \mathcal{A}^*(v)$ for any v not belonging to $K \cup (-K)$; hence $F_3(I) \in \mathcal{C}^*$. Moreover, the final argument of the previous example shows that $F_3(I)$ cannot be written as $\bigcup_{n=1}^{\infty} B_n$, where $B_n \in \mathcal{A}^*(v_n)$ and $v_n \in K$. Since there are complete sequences v_n consisting of elements of K, this shows that $F_3(I) \notin \mathcal{A}^*$. ∎

5. Which sets have to be exceptional?

The above results contribute towards a general problem of finding a geometric description of the (Borel) σ-ideal $\mathcal{L}(X)$ of subsets of a separable Banach space X generated by the sets of Gâteaux non-differentiability of Lipschitz mappings of X into spaces with the Radon–Nikodym property. The current results concerning this question seem to be rather weak. It is not even known if the σ-ideal $\mathcal{L}_{\mathbb{R}}(X)$ generated by sets of Gâteaux non-differentiability of real-valued Lipschitz functions on X coincides with $\mathcal{L}(X)$. Moreover, if X is finite dimensional and $\dim(X) \geq 3$, it is not known if $\mathcal{L}(X)$ coincides with Lebesgue null sets. This is the case if $\dim(X) \leq 2$; the case when $\dim(X) = 1$ is classical and the case $\dim(X) = 2$ follows from a recent result of the first named author that for every Lebesgue null set $M \subset \mathbb{R}^2$ there is a Lipschitz mapping $g: \mathbb{R}^2 \to \mathbb{R}^2$ which is non-differentiable at all points of M. Interestingly enough, this implies that as long as $\dim(X) \geq 2$, there is a set $N \in \mathcal{L}_{\mathbb{R}}(X)$ for which one cannot find a real-valued Lipschitz function Gâteaux non-differentiable at all points of N. To see this, let π be a continuous linear projection on a two dimensional subspace V of X, $\kappa: \ell_2 \to X$ a continuous linear mapping with a dense image and $M \subset V$ a G_δ set of Lebesgue measure zero containing all lines passing through two distinct points with rational coordinates (in some fixed basis for V). We define $N = \pi^{-1}(M)$. Note that $\kappa^{-1}(N) = (\pi \circ \kappa)^{-1}(M)$ and so, since $\pi \circ \kappa$ is an open mapping, for every $u, v \in \ell_2$ and $\varepsilon > 0$ there are $u', v' \in \ell_2$ such that $\|u - u'\| + \|v - v'\| < \varepsilon$ and the line segment $[u', v']$ lies in $\kappa^{-1}(N)$. So, by a result of [Pr], every Lipschitz real-valued function on ℓ_2 is Fréchet differentiable at a point of $\kappa^{-1}(N)$. Whenever $f: X \to \mathbb{R}$ is Lipschitz, we use this for $f \circ \kappa$ to find a point $y \in \kappa^{-1}(N)$ at which it is Fréchet differentiable. It is now easy to verify that f is Gâteaux differentiable at $x = \kappa(y) \in N$; so N is not contained in the set of points of non-differentiability of a single Lipschitz real-valued function. However, as was mentioned above, there is a Lipschitz mapping $g: V \to \mathbb{R}^2$ which is Lipschitz and non-differentiable at the points of M. Hence $g \circ \pi$ is Gâteaux non-differentiable at all points of N, so $N \in \mathcal{L}_{\mathbb{R}}(X)$.

From the observations on distance functions made before the statement of Theorem 1 it immediately follows that every σ-directionally porous set belongs to $\mathcal{L}_{\mathbb{R}}$. In fact the following stronger observation due to Bernd Kirchheim holds. (We note in passing that the statement holds also for σ-porous sets and Fréchet derivative; for this the argument of [PT] together with the decomposition of the sum from the end of the following proof is sufficient. See [HMWZ].)

PROPOSITION 14: *If X is a separable Banach space and $A \subset X$ is σ-directionally porous, then there is a real-valued Lipschitz function f on X which is Gâteaux non-differentiable at any point of A.*

Proof: We first note that for every set $S \subset X$ there is a function $h: X \to [0, \infty)$ such that $\mathrm{Lip}(h) \leq 4$, $\min(\mathrm{dist}(x, S), 1) \leq h(x) \leq 4 \min(\mathrm{dist}(x, S), 1)$ for all $x \in X$, and h is Gâteaux differentiable at every point of $X \setminus \overline{S}$. This may be seen as follows: Write

$$\mathrm{dist}(x, S) = \sum_{k=-\infty}^{\infty} g_k(x), \qquad \text{where } g_k(x) = \min(2^k, \max(\mathrm{dist}(x, S) - 2^k, 0))$$

and define $h_k = g_k \star \mu_k$, where μ_k is a non-degenerated cube measure (cf. [BL, p. 142]) with support in $B(0, 2^{k-2})$. Since g_k is clearly Lipschitz (with $\mathrm{Lip}(g_k) \leq 1$), it is Gâteaux differentiable at μ_k-almost all points of X (cf. [BL, 6.25, 6.27, 6.42]) and therefore Lebesgue's dominated convergence theorem easily gives (as in [BL, 6.43]) that $h_k'(x, v) = (g_k'(\cdot, v) \star \mu_k)(x)$ for every $x, v \in X$ and therefore h_k is an everywhere Gâteaux differentiable function.

It is easy to verify that $0 \leq h_k \leq 2^k$, $\mathrm{Lip}(h_k) \leq 1$, $|h_k - g_k| \leq 2^{k-2}$ and $h_k(x) = g_k(x)$ unless $2^k - 2^{k-2} < \mathrm{dist}(x, S) < 2^{k+1} + 2^{k-2}$. Letting $h = 2\sum_{k=-\infty}^{0} h_k$, we thus have $h(x) \leq 2\sum_{k=-\infty}^{0} 2^k = 4$, $h(x) = 0$ if $x \in \overline{S}$, $h(x) \geq 2h_0(x) \geq 2g_0(x) - 2^{-1} \geq 1$ if $\mathrm{dist}(x, S) \geq 2$, and, if $k \leq 0$ and $2^k \leq \mathrm{dist}(x, S) < 2^{k+1}$, then

$$h(x) \leq 2 \sum_{j=-\infty}^{k+1} (g_j(x) + 2^{j-2}) = 2\,\mathrm{dist}(x, S) + 2^{k+1} \leq 4\,\mathrm{dist}(x, S)$$

and

$$h(x) \geq 2 \sum_{j=-\infty}^{k} (g_j(x) - 2^{j-2}) = 2\,\mathrm{dist}(x, S) - 2^k \geq \mathrm{dist}(x, S).$$

Moreover, since $\|h_k'(x)\| \leq 1$ and $h_k'(x) = 0$ for all but two values of k (those for which $2^k - 2^{k-2} < \mathrm{dist}(x, S) < 2^{k+1} + 2^{k-2}$), we have that h is Gâteaux differentiable at every point of $X \setminus \overline{S}$ and $\|h'(x)\| \leq 4$; together with the inequality $0 \leq h(x) \leq 4\,\mathrm{dist}(x, S)$ this shows that $\mathrm{Lip}(h) \leq 4$.

Suppose now that A is a σ-directionally porous subset of X and write $A = \bigcup_n A_n$, where A_n is directionally porous. For each n use the above to find a function $h_n: X \to [0, \infty)$ such that $\mathrm{Lip}(h_n) \leq 4$, $\min(\mathrm{dist}(x, A_n), 1) \leq h_n(x) \leq$

$4\min(\operatorname{dist}(x, A_n), 1)$ for all $x \in X$, and h_n is Gâteaux differentiable at every point of $X \setminus \overline{A_n}$. Let $f_n = 2^{-n}h_n$ and $f = \sum_{n=1}^{\infty} f_n$. Clearly, f is Lipschitz on X. If $x \in A$, we write $f = f_m + \sum' f_n + \sum'' f_n$, where m is such that $x \in A_m$, the first sum extends over those $n \neq m$ for which $f_n(x) = 0$ and the second sum over those n for which $f_n(x) > 0$. Let v be the direction of porosity of A_m at x. The definition of porosity gives that

$$\limsup_{t \searrow 0} (f_m(x + tv) + f_m(x - tv) - 2f_m(x))/t > 0.$$

Since $f_n \geq 0$, we have $\sum'(f_n(x+tv) + f_n(x-tv) - 2f_n(x))/t \geq 0$ for every $t > 0$. Finally, the sum $\sum'' f_n$ is Gâteaux differentiable at x since all f_n involved in it are Gâteaux differentiable at x and $\sum \operatorname{Lip}(f_n) < \infty$. Hence

$$\lim_{t \searrow 0} \sum\nolimits'' (f_n(x + tv) + f_n(x - tv) - 2f_n(x))/t = 0,$$

and we conclude that $\limsup_{t\searrow 0}(f(x + tv) + f(x - tv) - 2f(x))/t > 0$, and so f is not Gâteaux differentiable at x. ∎

For any set $N \subset \mathbb{R}$ of measure zero and any projection π of X onto $\mathbb{R}$ the set $\pi^{-1}(N)$ belongs to $\mathcal{L}$; this is obvious by composing a function $f: \mathbb{R} \to \mathbb{R}$ non-differentiable at points of N with π. A more general version of this argument is given in the following statement.

PROPOSITION 15: *Suppose that X, Y are separable Banach spaces, $N \in \mathcal{L}(Y)$ and that $g: X \to Y$ is a Lipschitz mapping. For $x \in X$ denote by U_x the set of directions of (one sided) differentiability of g. Then the set of those $x \in g^{-1}(N)$ for which the span of the set $\{g'_+(x, u) : u \in U_x\}$ is dense in Y belongs to $\mathcal{L}(X)$.*

Proof: By Theorem 5, there is a σ-directionally porous set $A \subset X$ such that for every $x \in X \setminus A$ the set U_x is a closed linear subspace of X and the mapping $u \to g'_+(x, u)$ is linear on U_x.

By definition of $\mathcal{L}(Y)$ we can suppose without any loss of generality that there exists a Lipschitz function $f: Y \to Z$, where Z has the Radon–Nikodym property, which is Gâteaux non-differentiable at any point of N. If $x \in g^{-1}(N) \setminus A$, it is easy to see that $f \circ g$ is Gâteaux non-differentiable at x. Indeed, suppose to the contrary that $f \circ g$ is Gâteaux differentiable at x and consider a vector $u \in U_x$. An easy argument then shows that $f'(g(x), g'(x, u))$ exists and is equal to $(f \circ g)'(x, u)$ (this can be also deduced from Lemma 11 (iv)). Since $\{g'_+(x, u) : u \in U_x\}$ is a linear dense subspace of Y, we easily deduce (cf. [BL, Lemma 6.40]) that f is Gâteaux differentiable at x, which is a contradiction. Hence $g^{-1}(N) \setminus A$ belongs to $\mathcal{L}(X)$; since A belongs to $\mathcal{L}(X)$ by Proposition 14, this finishes the proof. ∎

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